MMP Learning Seminar

Week 49.

Content:

Existence of log canonical closures.

Proof of the Corollaries.

Existence of log canonical closures:

Theorem 1.1: $f : X \longrightarrow U$ projective morphism of normal varieties, Δ a $\mathbb{Q}$-divisor s.t (X, Δ) dlt and $S = \lfloor \Delta \rfloor$ the non-klt locus. Assume there exists $U^\circ \subseteq U$ for which $(X^\circ, \Delta^\circ) = (X, \Delta) \times_U U^\circ$ has a good minimal model over U° <u>any stratum of S intersects</u> X°. Then (X, Δ) has a good minimal model over U.

Canonical bundle formula:

Theorem: Let (X, Δ) be a dlt pair and $f: X \to U$ a projective morphism over a normal variety U.
Then there exists a commutative diagram:

$$\begin{tikzcd} X' \arrow[r, "\mu"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & U \end{tikzcd}$$

with the following properties:

a) μ birational, h equidimensional fibration, X' $\mathbb{Q}$-factorial toroidal, Y smooth $\leftarrow$ semistable reduction

b) (X', Δ') has toroidal singularities,

$$\mu_* \mathcal{O}_{X'}(m(K_{X'} + \Delta')) \simeq \mathcal{O}_X(m(K_X + \Delta)) \quad \forall m.$$

c) J g-nef over U, $B \geq 0$ s.t. we can write

$$h_* \mathcal{O}_{X'}(dm(K_{X'} + \Delta')) \simeq \mathcal{O}_Y(dm(K_Y + J + B)).$$

Canonical bundle formula:

(X,Δ) lc pair. $\qquad$ $(K_X+\Delta)$-MMP

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d] & X' \arrow[d, "\varphi"] \\ U & Y \arrow[l] \end{tikzcd}$$

$K_{X'}+\Delta'$ semiample over U.

$K_{X'}+\Delta' \sim_{\mathbb{Q},Y} 0.$

φ is a log Calabi-Yau fibration.

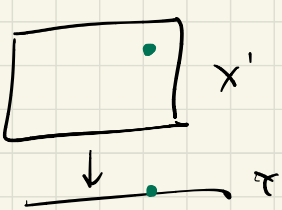
$$K_{X'}+\Delta' \sim_{\mathbb{Q}} \varphi^*(L_Y) \quad , \quad L_Y \text{ ample } \mathbb{Q}\text{-divisor.}$$

Q: Can we write L_Y as a log pair?

$L_Y = K_Y + \Delta_Y$ in some meaningful way?

Expectation: We can find a lc pair (Y,Δ_Y)

such that $K_{X'}+\Delta' \sim_{\mathbb{Q}} \varphi^*(K_Y+\Delta_Y)$



related log canonical centers.

Canonical bundle formula:

$$\begin{array}{c} X' \\ \varphi \big\downarrow \\ Y \end{array} \qquad K_{X'} + \Delta' \sim_{\mathbb{Q},Y} 0$$

Then we can find J which is a nef b-divisor. $\Delta_Y \geq 0$ s.t (Y, Δ_Y) has lc sing and

$$K_{X'} + \Delta' \sim_{\mathbb{Q}} \varphi^*(K_Y + \Delta_Y + J)$$

measures the sing fibers in codimension 1 of φ.

measures variation in moduli of the fibers

Corollary 1.2 (Existence of lc closures):

Let U° be an open subset of a normal qp variety U. $f^\circ: X^\circ \longrightarrow U^\circ$ be a projective morphism and (X°, Δ°) be a log canonical pair.

Then, there exists $f: X \longrightarrow U$ projective and a lc pair (X, Δ) s.t. $X^\circ = X \times_U U^\circ$ is an open and $\Delta^\circ = \Delta|_{X^\circ}$.

Proof: $f^c: X^c \longrightarrow U$ projective. Δ^c = closure of Δ in X^c

$\overline{X} \xrightarrow{\pi} X^c$ log resolution of

$(X^c, \Delta^c \cup f^{c^{-1}}(U \setminus U^\circ))$ Write

$$(\pi|_{X^\circ})^*(K_{X^\circ} + \Delta^\circ) = K_{\overline{X}^\circ} + \overline{\Delta}^\circ - F^\circ$$

effective with no comon comp.

where $\overline{X}^\circ = \pi^{-1}(X^\circ)$.

We have a commutative diagram:

$$\begin{tikzcd}
\overline{X}^{\circ} \arrow[r] \arrow[d, "\pi|_{\overline{X}^{\circ}}"'] & \overline{X} \arrow[r, dashed] \arrow[d] & X \arrow[d] \arrow[dl] \\
X^{\circ} \arrow[r] & X^{c} \arrow[r, "f^{c}"] & U
\end{tikzcd}$$

$\overline{\Delta}$ = closure of $\overline{\Delta}^{\circ}$ in $\overline{X}$.

$(\overline{X}, \overline{\Delta})$ and the morphism $\overline{X} \longrightarrow X^{c}$ satisfies the assumption 1.1 over X°.

Therefore, we can take (X, Δ) to be the ample model of $(\overline{X}, \overline{\Delta})$ over X^{c}. We have that.

$$(X, \Delta) \times_U U^{\circ} = (X^{\circ}, \Delta^{\circ}).$$

$\square$

Semistable pairs & reduction:

A morphism $f:(X,\Delta) \longrightarrow U$ from a lc pair to a smooth curve is said to be semistable if for all $p \in U$, we have that $(X, \mathrm{supp}(\Delta) + X_p)$ is log smooth & $X_p = f^{-1}(p)$ reduced.

Semistable reduction: (Toroidal embeddings 73').

X normal over $\mathbb{C}$, $f: X \longrightarrow C$ flat to a smooth curve

Theorem: There exists a finite morphism $C' \longrightarrow C$ from C' smooth and a projective resolution $X'' \to X'_n$

$$X'_n = X \times_C C'$$

such that $X'' \xrightarrow{f''} C$ satisfies the following.

(1) $(f'')^*(c') \cup \mathrm{Ex}(g)$ is snc for every $c' \in C'$.

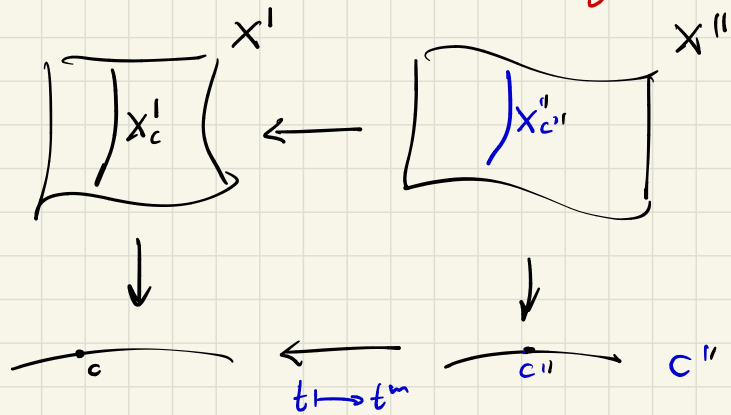
(2) $(f'')^*(c')$ is reduced for every $c' \in C'$.

Semistable reduction:

$$\begin{tikzcd} X \arrow[d] & X' \arrow[l, "\text{resolution}"'] \arrow[dl] \\ C & \end{tikzcd}$$

toroidal sing

$$f^*(c) = m X'_c$$



Corollary 1.4: U smooth curve.

$f^\circ : X^\circ \longrightarrow U$ affine finite type lc morphism

Then, there exists a finite dominant base change

$\theta : \tilde{U} \longrightarrow U$ and a projective morphism (lc)

$f : X \longrightarrow U$ s.t.:

$$X^\circ \times_U \tilde{U} \subseteq X \quad \text{and} \quad f|_{X^\circ \times_U \tilde{U}} = f^\circ \times_U \theta.$$

$$\begin{tikzcd} \tilde{X}^c \arrow[r] \arrow[d] & X^c \arrow[d] \\ \tilde{U} \arrow[r] & U \end{tikzcd}$$

Proof: $f : X^c \longrightarrow U$. $\theta : \tilde{U} \longrightarrow U$,

take a log resolution $\bar{X} \xrightarrow{\pi} (\tilde{X}^c, \tilde{X}^c \setminus \tilde{X}^\circ)$

$\bar{f} : (\bar{X}, Ex(\pi)) \longrightarrow \tilde{U}$ is semistable

Denote $\bar{X}^\circ = \pi^{-1}(\tilde{X}^\circ)$, $\tilde{X}^\circ$ has lc sing.

We have a commutative diagram:

$$\begin{tikzcd}
\overline{X}^\circ \arrow[r] \arrow[d, "\tilde{\pi}|_{\overline{X}^\circ}"'] & \overline{X} \arrow[r, dashed] \arrow[d, "\tilde{\pi}"'] & X \arrow[dl] \arrow[d] & \\
\tilde{X}^\circ \arrow[r] & \tilde{X}^c \arrow[r, "\tilde{f}^c"] & \tilde{U} \arrow[r, "\Theta"] & U.
\end{tikzcd}$$

Write $(\tilde{\pi}|_{X^\circ})^* K_{\tilde{X}^\circ} = K_{\overline{X}^\circ} + E - F$.

$\overline{\Delta} :=$ closure of E° in $\overline{X}$.

$(\overline{X}, \overline{\Delta})$ is a family of semistable pairs over $\tilde{U}$.

$(\overline{X}, \overline{\Delta} + \overline{X}_p)$ is dlt for every $p \in \tilde{U}$.

We can apply Theorem 1.1 to $(\overline{X}, \overline{\Delta})$ over $\tilde{X}^c$ where we choose the open set to be $\tilde{X}^\circ$.

$X = \mathrm{Proj}\, R(\overline{X}/\tilde{X}^c, K_{\overline{X}} + \overline{\Delta})$.

The induced map $X \longrightarrow \tilde{U}$ is an lc morphism.

$\tilde{X}^\circ$ is lc, then π is an isomorphism over $\tilde{X}^\circ$.

$\square$

Corollary 1.5: $f^\circ : X^\circ \longrightarrow U^\circ$ projective

(X°, Δ°) lc, $U \ni p$ is a germ of a smooth curve

$U \setminus \{p\} = U^\circ$, $K_{X^\circ} + \Delta^\circ$ is f-ample.

Then, there exists $\theta : \tilde{U} \longrightarrow U$ dominant

base change, (X, Δ) log canonical,

$(X, \Delta) \longrightarrow \tilde{U}$ s.t $K_X + \Delta$ is ample over $\tilde{U}$

and the restriction of (X, Δ) to the preimage

$\theta^{-1}(U^\circ)$ is isomorphic to $(X^\circ, \Delta^\circ) \times_U \tilde{U}$.

Theorem 1.6: Let $f: X \longrightarrow U$ be a projective morphism between normal varieties, Δ' and Δ'' effective $\mathbb{Q}$-divisors on X such that:

1) $(X, \Delta' + \Delta'')$ is $\mathbb{Q}$-factorial lc,
2) (X, Δ'') is dlt, and
3) $K_X + \Delta' + \Delta'' \sim_{\mathbb{Q}, U} 0$.

$$\begin{cases} \Delta'' = \Delta_Y + S_1 + \dots + S_r \\ \Delta' = \Gamma_Y / m \end{cases}$$

Then, the MMP with scaling for $K_X + \Delta''$ over U that terminates either with a MFS or a gmm.

Idea: Δ' does not intersect the general fiber.

$R(X/U, K_X + \Delta'')$ is f.g.

$$\begin{tikzcd}
X \arrow[d] & X' \arrow[l] \arrow[r, dashed] & Z \arrow[r, dashed] \arrow[d] & Z_1' \arrow[r, dashed] \arrow[d] & Z_2' \arrow[r, dashed] \arrow[d] & \cdots \arrow[r, dashed] & Z_N' \arrow[d] \\
U & & Y \arrow[ll] \arrow[r, dashed] & Y_1' \arrow[r, dashed] & Y_i' \arrow[r, dashed] & \cdots \arrow[r, dashed] & Y_N'
\end{tikzcd}$$

$K_Y + B + J$

$K_{Z_N'} + \Delta_{Z_N'}$ semiample over U.

Corollary 1.8: Let $f: X \longrightarrow Z$ be a flipping contraction for a log canonical pair (X, Δ). Then the flip exists.

$$\psi^*(K_X+\Delta+\Gamma/m) = K_Y+\Delta_Y+\Gamma_Y/m + \overbrace{S_1+\dots+S_r}^{\text{divisorial non-klt center}} \sim_{\mathbb{Q},Z} 0.$$

$\mathbb{Q}$-factorial

$K_{Y'}+\Delta_{Y'}+S_1'+\dots+S_r'$ semiample over Z.

$$\begin{tikzcd}
(Y, \Delta_Y) \arrow[r, dashed] \arrow[d, "\psi", "\text{dlt modification}"'] & (Y', \Delta_{Y'}) \arrow[d] \\
(X, \Delta) \arrow[dr, "\rho=1"'] & (X^+, \Delta^+) \arrow[d] \\
& Z
\end{tikzcd}$$

$-(K_X+\Delta)$ ample over Z.

$\Gamma \in |-m(K_X+\Delta)|$

$(X, \Delta+\Gamma/m)$ lc

$$\psi^*(K_X+\Delta) = \underbrace{K_Y+\Delta_Y+S_1+\dots+S_r}_{\text{good minimal model over } Z.} \qquad \square$$